

Some colouring problems for unit-quadrance graphs

Le Anh Vinh
 Department of Mathematics
 Harvard University
 Cambridge, MA 02138
 avle@fas.harvard.edu

Abstract

The quadrance between two points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ is the number $Q(A_1, A_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Let q be an odd prime power and F_q be the finite field with q elements. The unit-quadrance graph D_q has the vertex set F_q^2 , and $X, Y \in F_q^2$ are adjacent if and only if $Q(A_1, A_2) = 1$. In this paper, we study some colouring problems for the unit-quadrance graph D_q and discuss some open problems.

1 Introduction

In [11], Wildberger introduces a remarkable new approach to trigonometry and Euclidean geometry by replace distance by quadrance and angle by spread, thus allowing the development of Euclidean geometry over any field. The following definition follows from [11].

Definition 1 *The **quadrance** $Q(A_1, A_2)$ between the points $A_1 = (x_1, y_1)$, and $A_2 = (x_2, y_2)$ in F_q^2 is the number*

$$Q(A_1, A_2) := (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

This approach motives some colouring problems for unit-quadrance graph over any field. Note that in usual 2-dimensional Euclidean space R^2 then the quadrance between A_1, A_2 is unit if and only if the distance between A_1, A_2 is unit.

Let q be an odd prime power and F_q be the finite field with q elements. The unit-quadrance graph D_q has the vertex set F_q^2 , and $X, Y \in F_q^2$ are adjacent if and only if $Q(A_1, A_2) = 1$. This graph (or so-called finite Euclidean graph) was also studied by Medrano et al in [6].

This paper aims to outline some results about colouring problems for unit-quadrance graphs and also discuss several interesting open questions.

2 The chromatic number

Let $\chi(F_q^2)$ be the chromatic number of graph D_q . In [7], the author showed that

Theorem 1 *Suppose that $q = p^n > 3$ where p is an odd prime number then*

$$1 + \frac{q - (-1)^{(q-1)/2}}{q^{1/2}} \leq \chi(F_q^2) \leq \frac{p^n + p^{n-1}}{2} = q(1/2 + o(1)).$$

For some primes q , the graph D_q contains no triangle. We have the following lemma.

Lemma 1 ([7]) *Let q be any prime of the form $q = 12k \pm 7$. Then D_q contains no triangle.*

Proof Suppose that D_q contains a triangle XYZ with $X = (m, n)$, $Y = (m + x, n + y)$ and $Z = (m + x + u, n + y + v)$ for some $m, n, x, y, u, v \in F_q$ then $x^2 + y^2 = u^2 + v^2 = (x+u)^2 + (y+v)^2 = 1$. This implies that $xu + yv = -1/2$. We have $(xu + yv)^2 + (xv - yu)^2 = (x^2 + y^2)(u^2 + v^2)$ so

$$(xv - yu)^2 = 1 - 1/4 = 3/4.$$

For $q = 12 \pm 7$ then 3 is not square in F_q . Hence we have a contradiction. This concludes the proof of the lemma. \square

From Theorem 1 and Lemma 1, if $q \equiv \pm 7 \pmod{12}$ is a prime number then D_q is a triangle-free graph with chromatic number $\chi(D_q) \geq q^{1/2}(1 + o(1))$. This give us a family of triangle-free graphs with arbitrary high chromatic numbers.

Theorem 2 *For $q \equiv \pm 7 \pmod{12}$ is a prime number then the unit-quadrance graph D_q of order q^2 is a triangle-free graph with chromatic number number*

$$\chi(D_q) \geq q^{1/2}(1 + o(1)).$$

The bound on the independence numbers follows from an estimate of the Lovász θ -function. This function, introduced by Lovász in [5], can be defined as follows. If $G = (V, E)$ is a graph, an orthonormal labeling of G is a family $(b_v)_{v \in V}$ of unit vectors in an Euclidean space so that if u and v are distinct non-adjacent vertices, then $b_u^t b_v = 0$, that is, b_u and b_v are orthogonal. The θ -number $\theta(G)$ is the minimum, over all orthonormal labelings b_v of G and over all unit vectors c , of

$$\max_{v \in V} \frac{1}{(c^t b_v)^2}.$$

It is known that (see [5]) the independence number of G does not exceed $\theta(G)$. Theorem 9 of [5] asserts that for d -regular graphs G with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \dots \lambda_n$,

$$\theta(G) \leq \frac{-n\lambda_n}{\lambda_1 - \lambda_n}. \quad (1)$$

In [6], Medrano et al. give a general bound for eigenvalue of D_q .

Lemma 2 ([6]) *Let $\lambda \neq \Delta(D_q)$ be any eigenvalue of graph D_q then $|\lambda| \leq q^{1/2}$.*

Recall that the graph D_q is a regular graph with degree $\Delta(D_q) = q - (-1)^{(q-1)/2}$ (see [6]). From Lemma 2 and equation (1), we have

$$\theta(D_q) \leq \frac{-q^2(-q^{1/2})}{q - (-1)^{(q-1)/2} - (-q^{1/2})} \leq q^{3/2}. \quad (2)$$

Thus, the independence number of D_q does not exceed $q^{3/2}$. In other words, D_q contains no K_3 and \bar{D}_q contains no $K_{q^{3/2}}$. Therefore, we have a constructive bound for off-diagonal Ramsey number $R(m, 3)$.

Theorem 3 *For $q \equiv \pm 7 \pmod{12}$ is a prime number then the unit-quadrance graph D_q of order q^2 gives a constructive bound for off-diagonal Ramsey number*

$$R(m, 3) \geq \Omega(m^{2/3}).$$

This bound matches with the second best bound of Noga Alon in [1]. The problem of finding better bounds for the chromatic number of graph D_q touches on an important question in graph theory: what is the greatest possible chromatic number for a triangle-free regular graph of order n .

3 Perfect unit-quadrance graph

A graph G is perfect if and only if for every induced subgraph H of G , $\chi(H) = \omega(H)$ where $\omega(H)$ as usual denotes the clique number. We will determine for which q then the unit-quadrance graph D_q is perfect. We will need some properties of universal geometry.

Definition 2 *A circle in a finite field F_q with center $A_0 = [x, y]$ and quadrance $K \in F_q$ is set of all points $X = (u, v)$ in $F_q \times F_q$ such that*

$$Q(A_0, X) = K.$$

Let $C_i(X)$ denote the circle centered at $X \in F_q^2$ with quadrance i . In [10], the author proved the following result.

Lemma 3 ([10]) *For any $i, j \neq 0$ in F_q . Let X, Y be two distinct points in F_q^2 such that $k = Q(X, Y) \neq 0$. Then the number of intersections of two circles $C_i(X), C_j(Y)$ only depends on i, j and k . Precisely, define*

$$f(i, j, k) := ij - \frac{(i - j - k)^2}{4}.$$

The number of intersection points is p_{ij}^k , where

$$p_{ij}^k = \begin{cases} 0 & \text{if } f(i, j, k) \text{ is non-square,} \\ 1 & \text{if } f(i, j, k) = 0, \\ 2 & \text{if } f(i, j, k) \text{ is square.} \end{cases} \quad (3)$$

Proof Suppose that $X = [m, n]$ and $Y = [m + x, n + y]$ for some $m, n, x, y \in F_q$ then $x^2 + y^2 = k$. Suppose $C_i(X)$ intersects $C_j(Y)$ at Z where $Z = [m + x + u, n + y + v]$ for some $u, v \in F_p$. Then we have $u^2 + v^2 = j$ and $(x + u)^2 + (y + v)^2 = i$. It implies that

$$xu + yv = \frac{i - j - k}{2}.$$

But we have $(xu + yv)^2 + (xv - yu)^2 = (x^2 + y^2)(u^2 + v^2)$ so

$$(xv - yu)^2 = kj - \frac{(i - j - k)^2}{4} = ij - \frac{(k - i - j)^2}{4} = f(i, j, k).$$

If $f(i, j, k)$ is non-square number in F_q then it is clear that there does not exist such x, y, u, v , or $p_{ij}^k = 0$. Otherwise, let $\alpha = (i - j - k)/2$ and $f(i, j, k) = \beta^2$ for $0 \leq \beta \leq (p + 1)/2$ then

$$\begin{aligned} xv - yu &= \pm\beta, \\ xu + yv &= \alpha. \end{aligned}$$

Solving for (u, v) with respect to (x, y) we have

$$\begin{aligned} u &= (\alpha x \mp \beta y)/j, \\ v &= (\pm\beta y + \alpha x)/j. \end{aligned}$$

If $\beta = 0$ then we have only one (u, v) for each (x, y) , but if $\beta \neq 0$ then we have two pairs (u, v) . It implies (3) and concludes the proof of the lemma. \square

We are now ready to prove the main result of this section.

Theorem 4 D_q is not perfect except for D_3 .

Proof D_3 is perfect because D_3 is the line graph of $K_{3,3}$ and line graphs of bipartite graphs are perfect.

From Lemma 1, D_5 and D_7 have no triangle. So $\omega(D_5) = \omega(D_7) = 2$. Since D_q contains an odd cycle of length q for all q odd, $\chi(D_q) \geq 3$ for all $q \geq 5$. Thus, we have

$$\omega(D_5) \neq \chi(D_5), \omega(D_7) \neq \chi(D_7).$$

This implies that D_5, D_7 are not perfect.

For $q = 9$, then from Theorem 1, we have $\chi(D_9) \geq 4$. From Lemma 3, $\omega(D_9) \leq 3$ (and it is indeed that $\omega(D_9) = 3$). Thus $\omega(D_9) < \chi(D_9)$ or D_9 is not perfect.

For $q \geq 11$ then from Theorem 1, we have

$$\chi(D_q) \geq 1 + \frac{q - 1}{\sqrt{q}} > 4,$$

so $\chi(D_q) \geq 5$. From Lemma 3, $\omega(D_q) \leq 4$ for all q . Thus, $\omega(D_q) < \chi(D_q)$ for $q \geq 11$. This implies that D_q is not perfect for $q \geq 11$. This concludes the proof. \square

4 Some open problems

4.1 The choice number

A k -list-assignment L to the vertices of a graph G is the assignment of a list, $L(v)$, of at least k colours to every vertex v of G . The graph G is k -choosable if for every k -list-assignment, we can choose a colour for each vertex from its list such that no two adjacent vertices have the same colours. Then the choice number $ch(G)$ of G is the smallest number k such that G is k -choosable. For any graph G we have $\lceil n/\alpha(G) \rceil \leq \chi(G) \leq ch(G)$. We have the following theorem.

Theorem 5 *Given $\gamma > 0$, there exists a $q_0(\gamma)$ such that for every $q \geq q_0(\gamma)$ we have*

$$\chi(D_q) \leq ch(D_q) \leq \frac{(1 + \gamma)q^2}{\log_2 q}.$$

The proof of this theorem is omitted as it is similar to the proof of Theorem 3 in [3]. This theorem gives us a better upper bound for $\chi(D_q)$. The question of finding a better bound or a constructive colouring which matches with this bound is still open.

4.2 The achromatic number

The achromatic number $\psi(G)$ of a graph G is the greatest number of colours in a vertex colouring of G such that no two adjacent vertices have the same colour and for any pair of colours, there is at least one edge of G whose endpoints are coloured with this pair of colours. Suppose that we partition $V(D_q)$ into m classes then at least one class has at most q^2/m vertices. Each vertex in this class has degree $\Delta(D_q)$ and there is at least one edge of G whose endpoints are coloured with this pair of colours so

$$\frac{q^2}{m} \Delta(D_q) \geq m - 1.$$

This implies that $\psi(D_q) \leq q^{3/2}$. For the lower bound, it is easy to see that $\psi(D_q) \geq q + 1$ (since no two adjacent vertices have the same colour). These bounds however are still far from the truth.

4.3 Edge and total colouring

An edge colouring of a graph G is an assignment of colours to its edges so that no two incident edges have the same colour. The edge-chromatic number $\chi'(G)$ of a graph G is the least number k of colours for which G has an edge colouring with exactly k colours. It is well known that if G is a regular graph of common degree Δ with odd order then $\chi'(G) = \Delta + 1$. Thus, $\chi'(D_q) = q - (-1)^{(q-1)/2}$. The edge choice number $ch'(G)$ of a graph G is the smallest integer k such that whenever every edge of G is given a list of at least k colours, there exists an edge colouring of G in which every edge receives a colour from

its own list and no two incident edges have the same colour. The List Edge Colouring Conjecture states that for every graph G , $ch'(G) = \chi'(G)$. The confirmation or rejection of this conjecture for unit-quadrance graph is also an interesting problem.

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